Optimal bond portfolios with fixed time to maturity

Patrik Andersson

Ritsumeikan University
Department of Mathematical Sciences
patrik@ndersson.nu

Joint with A.N. Lagerås, AFA Insurance

CREST and 4th Ritsumeikan-Florence Workshop
March 10, 2012

Outline

- Interest rates and rolling horizon bonds
- Affine term structure
- Mean variance portfolio
- Case study

Motivation

- Consider a pension fund that has guaranteed a certain future payment
- To hedge this liability the fund buys bonds with matching duration
- If the fund is in steady state the duration of the liabilities is constant in time
- It is then natural to consider bond investments with constant duration
- Also for other investors it is natural to think in terms of what durations to choose.

Rolling horizon bonds

- Zero-coupon bond: Contract that pays $ 1 at time of maturity
- Rolling horizon bond: Start with 1 zero-coupon bond with time to maturity τ. At all future times, rebalance to hold only z-c bonds with time to maturity τ.
- Considering RH bonds simplifies the analysis of bond portfolios

Value of a rolling horizon bond I

- Let $f_t(\tau', \tau'')$ be the simple forward rate contracted at time t for the period $t + \tau'$ to $t + \tau''$
- Let $f_t(\tau)$ be the instantaneous forward rate at time t for maturity $t + \tau$.
- The price of a z-c bond is

$$\mathcal{Z}_t(\tau) = \prod_{i=1}^{n} (1 + (\tau_i - \tau_{i-1}) f_t(\tau_{i-1}, \tau_i))^{-1} = \exp\left\{ - \int_0^{\tau} f_t(u) du \right\}$$

Value of a rolling horizon bond I

- Let $f_t(\tau', \tau'')$ be the simple forward rate contracted at time t for the period $t + \tau'$ to $t + \tau''$
- Let $f_t(\tau)$ be the instantaneous forward rate at time t for maturity $t + \tau$.
- The price of a z-c bond is

$$\mathcal{Z}_t(\tau) = \prod_{i=1}^{n} (1 + (\tau_i - \tau_{i-1}) f_t(\tau_{i-1}, \tau_i))^{-1} = \exp\left\{ - \int_0^{\tau} f_t(u) du \right\}$$

- If the rebalancing is in discrete time the value of a RH bond is

$$R^d_{t_i}(\tau) = \frac{\mathcal{Z}_{t_i}(\tau)}{\mathcal{Z}_0(\tau)} \prod_{k=1}^{i} \left(1 + \Delta_k f_{t_k}(\tau - \Delta_k, \tau)\right), \quad i = 0, 1, \ldots$$

Value of a rolling horizon bond II

- If the rebalancing is in continuous time

$$R_t(\tau) = \frac{\mathcal{Z}_t(\tau)}{\mathcal{Z}_0(\tau)} \exp\left\{ \int_0^t f_s(\tau) ds \right\}, \quad t \geq 0.$$

Value of a rolling horizon bond II

- If the rebalancing is in continuous time

$$R_t(\tau) = \frac{\mathcal{Z}_t(\tau)}{\mathcal{Z}_0(\tau)} \exp\left\{ \int_0^t f_s(\tau) ds \right\}, \quad t \geq 0.$$

- Since

$$\mathcal{Z}_t(\tau) = \exp\left\{ -\int_0^\tau f_t(u) du \right\},$$

we have

$$\begin{aligned} \log R_t(\tau) &= -\int_0^\tau (f_t(u) - f_0(u)) du + \int_0^t f_s(\tau) ds \\ &= \text{shift of forward curve} \; + \; \text{accumulated forward rate} \end{aligned}$$

Affine term structure

- The relation between interest rate and time to maturity is called yield curve or *term structure*
- If $f_t(\tau) = \kappa(\tau)' F_t$, we say that the term structure is affine

Affine term structure

- The relation between interest rate and time to maturity is called yield curve or *term structure*
- If $f_t(\tau) = \kappa(\tau)' F_t$, we say that the term structure is affine
- Example: Vasiček

$$f_t(\tau) = \mu \frac{1 - e^{-\lambda\tau}}{\lambda} - \frac{\sigma^2}{2} \left(\frac{1 - e^{-\lambda\tau}}{\lambda} \right)^2 + e^{-\lambda\tau} r_t$$

Nelson-Siegel

$$f_t(\tau) = \beta_{0t} + e^{-\gamma\tau} \beta_{1t} + \gamma\tau e^{-\gamma\tau} \beta_{2t}.$$

Principal component analysis

Rolling horizon bonds in an affine model

- In an affine model:

$$\log R_t(\tau) = -\bar{\kappa}(\tau)'(F_t - F_0) + \kappa(\tau)'\bar{F}_t,$$

where $\bar{F}_t = \int_0^t F_s ds$, $\bar{\kappa}(\tau) = \int_0^\tau \kappa(u)du$

Rolling horizon bonds in an affine model

- In an affine model:

$$\log R_t(\tau) = -\bar{\kappa}(\tau)'(F_t - F_0) + \kappa(\tau)'\bar{F}_t,$$

  where $\bar{F}_t = \int_0^t F_s ds$, $\bar{\kappa}(\tau) = \int_0^\tau \kappa(u)du$
- A portfolio of RH bonds, never rebalanced, has value

$$\Pi_t^0 = \sum_i \nu_i R_t(\tau_i) = \sum_i \nu_i \exp\{-\bar{\kappa}(\tau_i)'(F_t - F_0) + \kappa(\tau_i)'\bar{F}_t\}.$$

Generalized Ornstein-Uhlenbeck processes

- We will assume that F_t is a Generalized OU process
- Let L_t be a process with stationary independent increments and

$$F_t \equiv e^{-\lambda t} F_0 + \int_0^t e^{-\lambda(t-s)} dL_s$$

- F is then a Generalized OU process.

Generalized Ornstein-Uhlenbeck processes

- We will assume that F_t is a Generalized OU process
- Let L_t be a process with stationary independent increments and

$$F_t \equiv e^{-\lambda t} F_0 + \int_0^t e^{-\lambda(t-s)} dL_s$$

- F is then a Generalized OU process.
- Define the cumulant generating function

$$I(\theta \ddagger L_1) = \log E[e^{\theta L_1}] \equiv \zeta(\theta).$$

- Examples of ζ:
  $L_1 \sim N(\mu, \sigma)$: $\zeta(\theta) = \mu\theta + \sigma^2\theta^2/2$
  $L_1 \sim \Gamma(\nu, \alpha) : \zeta(\theta) = -\nu \log(1 - \theta/\alpha)$
  $F_\infty \sim N(\mu, \sigma) : \zeta(\theta) = \lambda\mu\theta + \lambda\sigma^2\theta^2$

Mean variance portfolio I

- Assume that the forward rate for each tradeable maturity is

$$f_t(\tau) = \kappa(\tau)^T F_t + e_t(\tau),$$

  where F_t is GOU with independent components and $e_t(\tau)$ has a stationary distribution, independent of F_t.

Mean variance portfolio I

- Assume that the forward rate for each tradeable maturity is

$$f_t(\tau) = \kappa(\tau)^T F_t + e_t(\tau),$$

  where F_t is GOU with independent components and $e_t(\tau)$ has a stationary distribution, independent of F_t.
- Then

$$\log R_\Delta(\tau) = -\bar{\kappa}(\tau)'(F_\Delta - F_0) + \kappa(\tau)'\bar{F}_\Delta + \bar{e}_\Delta(\tau),$$

  where we set the integrated residual

$$\bar{e}_\Delta(\tau) = -\int_0^\tau (e_\Delta(u) - e_0(u))du + \int_0^\Delta e_s(\tau)ds.$$

Mean variance portfolio II

- The expected return is then

$$\mathrm{E}\left[R_\Delta(\tau)\right] \\ = \mathrm{E}\Big[\exp\Big\{-\bar{\kappa}(\tau)'(F_\Delta - F_0) + \kappa(\tau)'\bar{F}_\Delta\Big\}\Big]\mathrm{E}\Big[\exp\{\bar{e}_\Delta(\tau)\}\Big]$$

Mean variance portfolio II

- The expected return is then

$$\begin{aligned} &\mathrm{E}\left[R_\Delta(\tau)\right] \\ &= \mathrm{E}\Big[\exp\big\{-\bar{\kappa}(\tau)'(F_\Delta - F_0) + \kappa(\tau)'\bar{F}_\Delta\big\}\Big]\mathrm{E}\Big[\exp\{\bar{e}_\Delta(\tau)\}\Big] \end{aligned}$$

- The first expectation can be written

$$\exp\left\{\sum_{i=1}^{d}\left(\lambda_i\bar{\kappa}_i(\tau) + \kappa_i(\tau)\right)\varepsilon(\Delta;\lambda)F_{i,0} + S_\tau\right\},$$

where $\varepsilon(t;\lambda) \equiv (1 - e^{-\lambda t})/\lambda$ and

$$S_\tau = \sum_{i=1}^{d}\int_0^\Delta \zeta_i\big(-\bar{\kappa}_i(\tau)e^{-\lambda_i s} + \kappa_i(\tau)\varepsilon(s;\lambda_i)\big)ds.$$

Estimation I

- Remains to estimate λ, ζ, $\mathrm{E}\left[\exp\left\{\bar{e}_\Delta(\tau)\right\}\right]$ and $\mathrm{E}\left[\exp\left\{\bar{e}_\Delta(\tau_1)\right\} + \bar{e}_\Delta(\tau_2)\right]$

Estimation I

- Remains to estimate λ, ζ, $\mathrm{E}\left[\exp\left\{\bar{e}_{\Delta}(\tau)\right\}\right]$ and $\mathrm{E}\left[\exp\left\{\bar{e}_{\Delta}(\tau_1)\right\} + \bar{e}_{\Delta}(\tau_2)\right]$
- λ and ζ:
  Parametric: Choose family of distributions and do likelihood or LS estimation.

Estimation I

- Remains to estimate λ, ζ, $\mathrm{E}\left[\exp\left\{\bar{e}_\Delta(\tau)\right\}\right]$ and $\mathrm{E}\left[\exp\left\{\bar{e}_\Delta(\tau_1)\right\} + \bar{e}_\Delta(\tau_2)\right]$
- λ and ζ:
  Parametric: Choose family of distributions and do likelihood or LS estimation.
  Non-parametric: $\zeta(s) = \sum_{n=1}^{\infty} \frac{c^n}{n!} s^n$, where c_n are the cumulants of L_1.

$$\int_0^\Delta \zeta\big(- \bar{\kappa} e^{-\lambda s} + \kappa\varepsilon(s;\lambda)\big) ds$$
$$= \sum_{n=1}^{\infty} \frac{c_n}{n!} \frac{1}{\lambda^n} \left[\sum_{k=0}^{n} \binom{n}{k} (-\lambda\bar{\kappa} - \kappa)^{n-k} \kappa^k \varepsilon\left(\Delta, \lambda(n-k)\right)\right]$$

Estimation II

- However we do not observe L_1 directly

$$F_{t+\Delta} = e^{-\lambda\Delta}F_t + \int_t^{t+\Delta} e^{-\lambda(t+\Delta-s)}dL_s \equiv \mathrm{E}\left[\epsilon_\Delta\right] + aF_t + (\epsilon_\Delta - \mathrm{E}\left[\epsilon_\Delta\right]).$$

Estimation II

- However we do not observe L_1 directly

$$F_{t+\Delta} = e^{-\lambda\Delta} F_t + \int_t^{t+\Delta} e^{-\lambda(t+\Delta-s)} dL_s \equiv \mathrm{E}\left[\epsilon_\Delta\right] + aF_t + (\epsilon_\Delta - \mathrm{E}\left[\epsilon_\Delta\right]).$$

- Thus, regressing $F_{t+\Delta}$ on F_t we get estimates $\widehat{\mathrm{E}\left[\epsilon_\Delta\right]}$, $\hat{a}$ and $\hat{\epsilon}_\Delta$, and we can set $\hat{\lambda} = -\frac{\log \hat{a}}{\Delta}$.

Estimation II

- However we do not observe L_1 directly

$$F_{t+\Delta} = e^{-\lambda\Delta}F_t + \int_t^{t+\Delta} e^{-\lambda(t+\Delta-s)}dL_s \equiv \mathrm{E}\left[\epsilon_\Delta\right] + aF_t + (\epsilon_\Delta - \mathrm{E}\left[\epsilon_\Delta\right]).$$

- Thus, regressing $F_{t+\Delta}$ on F_t we get estimates $\widehat{\mathrm{E}\left[\epsilon_\Delta\right]}$, $\hat{a}$ and $\hat{\epsilon}_\Delta$, and we can set $\hat{\lambda} = -\frac{\log \hat{a}}{\Delta}$.
- Also,

$$I(\theta \ddagger \epsilon_\Delta) = I\left(\theta \ddagger \int_0^\Delta e^{-\lambda s}dL_s\right) = \int_0^\Delta \zeta(\theta e^{-\lambda s})ds$$

$$= \int_0^\Delta \sum_{n=1}^{\infty} \frac{c_n}{n!}\theta^n e^{-\lambda ns}ds = \sum_{n=1}^{\infty} \frac{c_n}{n!}\theta^n \varepsilon(\Delta, \lambda n).$$

Estimation III

- We then have as a natural estimator

$$\hat{c}_n(L_1) = \frac{\hat{c}_n(\epsilon_\Delta)}{\varepsilon(\Delta, \hat{\lambda} n)},$$

Estimation III

- We then have as a natural estimator

$$\hat{c}_n(L_1) = \frac{\hat{c}_n(\epsilon_\Delta)}{\varepsilon(\Delta, \hat{\lambda} n)},$$

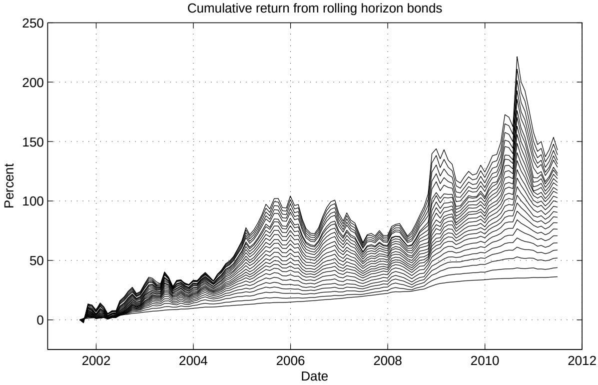
- $\mathrm{E}\left[\exp\left\{\bar{e}_{\Delta t}(\tau)\right\}\right]$: Estimate non-parametrically by replacing $\int \to \sum$ and thus approximating $\bar{e}_{\Delta t_i}(\tau_j)$ from data.

Case study I: European swap rates

- From monthly data on european swap rates we calculate the corresponding zero coupon rates with yearly spaced time to maturity between 1 and 25 years.

Case study II

- We calculate the return from a monthly rolling horizon bond.
- Yields for maturities not in the data are obtained by linear interpolation.



- Volatility and return are increasing in time to maturity

Case study III: Modelling

- We fit a Nelson-Siegel function to each yield curve, obtaining 3 time series.

$$y_t(\tau) = \beta_{0t} + \beta_{1t}\left(\frac{1 - e^{-\gamma\tau}}{\gamma\tau}\right) + \beta_{2t}\left(\frac{1 - e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau}\right).$$

- The root mean squared errors are of the order a couple of basis points
- We estimate, using ML, independent OU-Normal processes for these time series

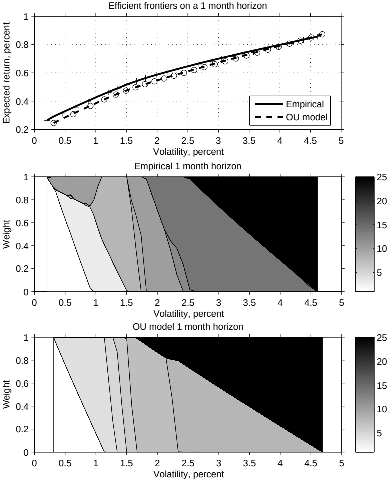
Case study III: Modelling

- We fit a Nelson-Siegel function to each yield curve, obtaining 3 time series.

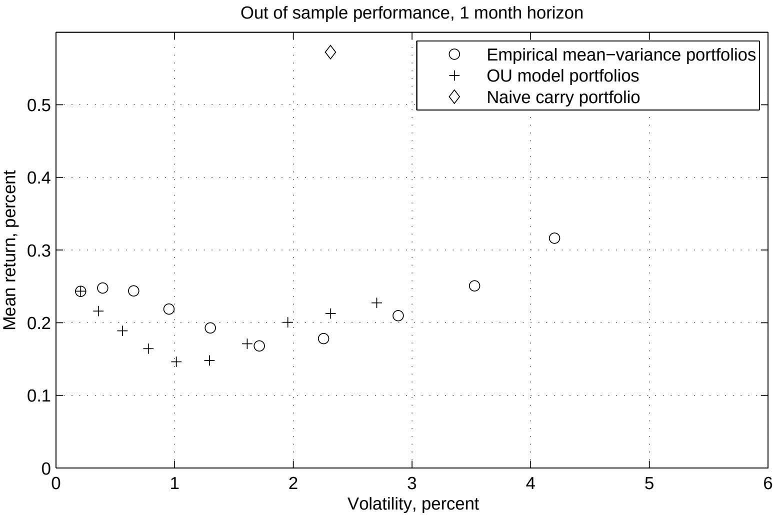
$$y_t(\tau) = \beta_{0t} + \beta_{1t}\left(\frac{1-e^{-\gamma\tau}}{\gamma\tau}\right) + \beta_{2t}\left(\frac{1-e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau}\right).$$

- The root mean squared errors are of the order a couple of basis points
- We estimate, using ML, independent OU-Normal processes for these time series
- We may then calculate the mean-variance optimal portfolio of rolling horizon bonds
- We also calculate an ”empirical” portfolio based directly on the observed returns

Case study IV: Mean-variance portfolio (no short selling, normal initial yield curve)



Case study V: Out of sample performace



Case study VI: Out of sample performace